PROPAGATION OF STATISTICS IN SYSTEMS

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Summary

A structure comparable to a "transfer function" for statistics in linear systems is developed so that, given a system input characteristic function, the high-order output charac-function is directly expressible as products of functions similar to that of the input. Emphasis has been placed on samples and sampled-data systems because sampled signals are easy to describe statistically. The results apply to the continuous case when the sampling rate is made high enough to satisfy the Nyquist sampling theorem.

The "memory" of sampled-data filters causes output sequences to be high-order processes. Distinctions are drawn between those which are Markov and those which are other kinds of high-order processes.

Introduction

In many system analysis and synthesis problems, it has been found that a statistical point of view gives averages which contain most of the important features of situations, and are much fewer in number than the number of situations from which they are taken. This paper is concerned with the following: Given the statistics of a random stationary signal at the input of a system, find those of the system output. The problem area will be restricted to dealing with linear sampled-data systems. Of interest are moments and distribution densities (and their Fourier transforms, or characteristic functions). A system model will be sought that will serve as a "transfer function" for statistics.

The statistical characteristics of a sampled signal may be clearly defined by a joint probability density distribution, where the joint probability density is a function of as many variables as there are samples in a row that are statistically related. This number will be called the order of the random process. This order becomes infinite for a continuous signal, which may be considered the limiting case of a sampled signal. It can be shown that the complete statistical description of a continuous signal or any statistical aspect of same can in principle be derived from the joint probability density of its samples as long as the Nyquist sampling theorem was satisfied when taking the samples. It is thus possible to assign an order to a continuous process equal to that of its samples when the sampling theorem is just satisfied. In any event, it is much simpler and more practical to deal with statistics of sampled data than continuous. Actually, knowledge of what happens every now and then is usually adequate when one is only interested in statistics.

A linear sampled-data system gives output samples at regular "sample times" which are a linear combination of the present and the past input samples. The coefficients of this linear combination make up the weighing function or the unit impulse response, which completely describes the system. An alternative system description is the transfer function, the transform of the impulse response. The transfer functions of sampled-data systems are always periodic in the "S-plane," repeating for every increment to real frequency of the radian sampling frequency. The frequency-domain description will not be used here, as the impulse response idea is the better approach to statistical propagation.

Before deriving system output characteristic functions and distributions, it would be well to take a detour and discuss the first and second moments of this output signal. These are its mean and mean square.

Derivation of Output Moments

The mean of the output is the mean of the input multiplied by the sum of the impulses of the unit impulse response. This is true regardless of the order of the input. When the input is first order (all input samples statistically independent of each other, and average is zero), the mean square of the output is the mean square of the input multiplied by the sum of the squares of the impulses of the impulse response; again, when the input is first order, the mean cube (third moment) of the output is that of the input multiplied by the sum of the cubes of the impulses of the unit impulse response. Any odd moment may be found in like manner provided that the input is first order. Higher even moments may be found with a slight modification of this procedure.

An example of a sampled-data system driven by a random input is shown in Figure 1. The average output response sample is

$$\begin{aligned} \bar{r} &= \bar{x}(1 + 1/2 + 1/4 + \cdots) \\ &= \bar{x}\left(\frac{1}{1-1/2}\right) = \bar{x}(2) \end{aligned} \tag{1}$$

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If the system of Figure 1 is a sampled "equivalent" of a continuous system, the size of the average output impulse is a measure of the average of the continuous envelope. If the input is first order, the mean square of the output is

$$\begin{aligned}\overline{r^2} &= \overline{x^2}\left(1+(1/2)^2+(1/4)^2+\cdots\right) \\ &= \frac{\overline{x^2}}{1-(1/2)^2} = \overline{x^2}\,(4/3)\end{aligned} \qquad (2)$$

Equation (2) comes from the idea that the filter output is a linear combination of past inputs, where each input is independent and has the same expected variance. The effective mean square contribution of a past input sample is its mean multiplied by the square of the factor by which it is weighed in forming the output. In like manner, the mean cube is

$$\begin{aligned}\overline{r^3} &= \overline{x^3}\left(1+(1/2)^3+(1/4)^3+\cdots\right) \\ &= \frac{\overline{x^3}}{1-(1/2)^3} = 8/7\,\overline{x^3}\end{aligned} \qquad (3)$$

First order random signals are quite common in systems. They are the easiest to deal with and could often approximately represent higher order processes. When the input is first order, the mean and variance of the output are very easy to get for linear sampled-data systems (all that is involved is summing geometric series) and are usually the most important features. If it is known that the output signal is Gaussian distributed and high order, the first order aspect of the output is completely determined by the mean and variance. The output will be Gaussian if the input is Gaussian. The output will be approximately Gaussian if the impulse response of the filter has about a half dozen or more impulses having magnitudes that are of the same order as the largest impulse magnitude. The authority for this is the Central Limit Theorem.

Derivations of Output Characteristic Functions for First-Order Inputs

The next subject to be considered, the main subject matter of this paper, is how to derive an exact expression for the characteristic function of the filter output given that of the input. This will be done for first-order inputs, and will be indicated for higher order inputs. The filter outputs will always be higher order than first. The discussion will commence, however, with the derivation of first order characteristic functions (c.f.) of the filter outputs.

First-Order Output Characteristic Functions

Consider the sampled-data system of Figure 2, as driven by a first-order input having a distribution density $W(x)$ and a c.f.

$$F_x(\xi) = \int_{-\infty}^{\infty} W(x)\, e^{-jx\xi}\, dx$$

The input signal is really $x(t)$, but the "signal" of interest here is $W(x)$ or its transform $F_x(\xi)$. The output is a linear combination of the present input and the three preceeding inputs, all being statistically independent for the first-order input. The output c.f. is as in equation (4).

$$F_r(\xi) = F_x(a\xi)\,F_x(b\xi)\,F_x(c\xi)\,F_x(d\xi) \qquad (4)$$

The c.f. of the sum of independent quantities is the product of their c.f.'s, and the c.f. of a quantity after a scale change by the factor "a" is $F_x(a\xi)$, if the original c.f. were $F_x(\xi)$. Thus, the first-order output c.f. will always be a product of c.f.'s if the input is first order. The output of the sampled-data filter of Figure 2, is a fourth-order process because the present output sample is statistically related to the three previous output samples. The next problem is to derive the fourth order c.f. of the output, given the first order c.f. of the input. It is necessary to take a detour here and develop a general mathematical "gimmic" for the calculation of the joint c.f. of statistically related variables.

Calculation of High Order Characteristic Functions

Consider the two first-order statistically related variables r_0 and r_{-1} shown in Figure 3. Sums of these variables will be first order processes whose c.f.'s will depend on their mutual relationship. It is possible to determine an arbitrary point on the two-dimensional c.f. by calculating the one-dimensional c.f. of an appropriate linear combination of the two variables.

By definition, the two-dimensional c.f. is given in terms of the two-dimensional distribution density $W(r_0, r_{-1})$ by the Fourier transform (5).

$$F_{r_0, r_{-1}}(\xi_0, \xi_{-1}) = \iint_{-\infty}^{\infty} W(r_0, r_{-1})\, e^{-j(r_0\xi_0 + r_{-1}\xi_{-1})}\, dr_0\, dr_{-1} \qquad (5)$$

The one-dimensional c.f. of a linear combination of r_o and r_{-1} ie, $(k_or_o + k_{-1}r_{-1})$ is given by equation (6).

$$F_\Sigma(\xi) = \iint_{-\infty}^{\infty} W(r_o, r_{-1})\, e^{-j(r_o k_o + r_{-1} k_{-1})\xi}\, dr_o\, dr_{-1} \tag{6}$$

Equations (5) and (6) are expressions for very different things which practically look alike. It follows that the value of $F_{r_o,r_{-1}}(\xi_o, \xi_{-1})$ can be obtained from $F_\Sigma(\xi)$ by "adjustment" of k_o and k_{-1} and a choice of ξ such that $k_o\xi = \xi_o$ and $k_{-1}\xi = \xi_{-1}$. This "gimmic" will be applied to the investigation of the statistical relation between adjacent (in time) samples at the output of a linear filter.

C.f. of Output of One Memory-State Filter

A simpler finite-memory filter than the one in Figure 2 results when the impulse response has only two impulses, ie., c = d = 0. The output equals "a" times the present input plus "b" times the previous input. The filter action is sequential, but it can be represented combinationaly by the flow graph of Figure 4.

In Figure 4, the "cards are laid on the table." It is possible to show present and past inputs and outputs in addition to how outputs are formed from inputs. The input nodes make up an "analog stepping register." A new input sample appears at the node "x_o" each sample time, while all the old input samples are indexed down by one node. All input samples are assumed to be statistically independent of each other and, since the input process is stationary, have the same statistical expectations.

It can be seen from Figure 4 that r_o and r_{-1} are statistically related, because they have something in common. On the other hand, r_o and r_{-2} have nothing in common, and because the input samples are first order, r_o and r_{-2} are unrelated. The output is second order, having an order one greater than the input because of the one memory-state of the filter. The second-order distribution $W(r_1,r_{-1})$ or its c.f. completely describes the output process.

$F_{r_o,r_{-1}}(\xi_o, \xi_{-1})$ will be derived by using the above "gimmic" and the construction of Figure 5, a modification of Figure 4. "Σ" is a linear combination of the independent signals x_o, x_{-1} and x_{-2}. Therefore it has the c.f. of equation (7), a product of three factors.

$$F_\Sigma(\xi) = F_x(k_o\xi)\, F_x\big[(ak_o + k_{-1})\xi\big]\, F_x(ak_{-1}\xi) \tag{7}$$

By making the substitutions of equations (8)

$$\left.\begin{aligned} k_o\xi &= \xi_o \\ k_{-1}\xi &= \xi_{-1} \end{aligned}\right\} \tag{8}$$

The desired $F_{r_o,r_{-1}}(\xi_o, \xi_{-1})$ results as equation (9).

$$F_{r_o,r_{-1}}(\xi_o,\xi_{-1}) = F_x(\xi_o)\, F_x(a\xi_o + \xi_{-1})\, F_x(a\xi_{-1}) \tag{9}$$

An example should do well to tie down this idea. For simplicity, let the input be Gaussian such that

$$\left.\begin{aligned} W(x) &= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}} \\ F_x(\xi) &= e^{-\frac{\xi^2\sigma^2}{2}} \end{aligned}\right\} \tag{10}$$

The two-dimensional output c.f. is given by the equations (11) by direct substitution in equation (9).

$$\begin{aligned} F_{r_o,r_{-1}}(\xi_o,\xi_{-1}) &= e^{-\frac{\xi_o^2\sigma^2}{2}}\, e^{-\frac{(a\xi_o+\xi_{-1})^2\sigma^2}{2}}\, e^{-\frac{(a\xi_{-1})^2\sigma^2}{2}} \\ &= e^{-\frac{\sigma^2}{2}\left[\xi_o^2(1+a^2) + \xi_{-1}^2(1+a^2) + 2a\xi_o\xi_{-1}\right]} \end{aligned} \tag{11}$$

This c.f. is two-dimensional Gaussian, having a mean square (second moment of first-order distribution) of $(1 + a^2)\sigma^2$ and a correlation coefficient of $\frac{a}{1+a^2}$. It is a general result that Gaussian input of any order will give a Gaussian output. This is true because higher order outputs will also appear as products of factors, and products of Gaussian factors are still Gaussian.

Third and Higher Order Output C.f.'s

The next more general problem is that of a filter having two memory states. The output is the present input plus "a" times the previous plus "b" times the next previous input. The appropriate construction for the calculation of $F_{r_o,r_{-1},r_{-2}}(\xi_o, \xi_{-1}, \xi_{-2})$ analogous to Figure 5 is Figure 6. The output can be seen to be third order. There is statistical connection between

r_o and r_{-1} because they have something in common. Likewise for r_o and r_{-2}, since they share a signal emanating from x_{-2}. This is not so for r_o and r_{-3} however.
From Figure 6

$$F_r(\xi) = F_x[k_o\xi]\,F_x[(ak_o+k_{-1})\xi]\,F_x[(bk_o+ak_{-1}+k_{-2})\xi]$$
$$F_x[(bk_{-2})\xi]\,F_x[(bk_{-1}+ak_{-2})\xi] \qquad (12)$$

By substituting

$$\left.\begin{aligned} k_o\xi &= \xi_o \\ k_{-1}\xi &= \xi_{-1} \\ k_{-2}\xi &= \xi_{-2} \end{aligned}\right\} \qquad (13)$$

$$F_{r_o,r_{-1},r_{-2}}(\xi_o,\xi_{-1},\xi_{-2})$$
$$= F_x(\xi_o)\,F_x(a\xi_o+\xi_{-1})\,F_x(b\xi_o+a\xi_{-1}+\xi_{-2})$$
$$F_x(b\xi_{-1}+a\xi_{-2})\,F_x(b\xi_{-2}) \qquad (14)$$

If for example the input is first-order Gaussian having c.f. of $e^{-\frac{\xi^2\sigma^2}{2}}$, the output c.f. will be the three-dimension Gaussian form of equation (15):

$$F_{r_o,r_{-1},r_{-2}}(\xi_o,\xi_{-1},\xi_{-2}) \qquad (15)$$
$$= e^{-\frac{\sigma^2}{2}\left[(\xi_o^2+\xi_{-1}^2+\xi_{-2}^2)(1+a^2+b^2) + (\xi_o\xi_{-1}+\xi_{-1}\xi_{-2})(2a+2b)+\xi_o\xi_{-2}(2b)\right]}$$

The mean square of the output is $\sigma^2(1 + a^2 + b^2)$ from equation (15). This agrees with the idea of equation (2). The correlation coefficient is $\frac{(a+b)}{(1+a^2+b^2)}$ between two adjacent samples and $\frac{b}{(1+a^2+b^2)}$ between first and third samples.

Equation (14) gives the three-dimensional output c.f. of a 2-memory-state filter. An expression for the (N + 1) - dimensional output c.f. of a N-memory-state filter can be easily induced from it. Let such a filter have an impulse response whose impulses are 1, a, b,....m, n. The desired c.f. has (2N + 1) factors, and is given by equation (16).

The c.f. of equation (16) is easy to obtain for a given filter, but it is not a convenient thing to deal with when the filter has many memory states and one requires the multidimensional distribution density. If one desires moments, however, they are obtained from the c.f. by differentiation; high-order moments come from high-order partial derivatives and joint moments come from cross partial derivatives, all evaluated at

$$\xi_o = \xi_{-1} = \cdots\cdots = \xi_{-N} = 0.$$

$$F_{r_o,r_{-1}\cdots r_{-N}}(\xi_o,\xi_{-1}\cdots\xi_{-N})$$
$$= F_x(\xi_o)\,F_x(a\xi_o+\xi_{-1})\,F_x(b\xi_o+a\xi_{-1}+\xi_{-2})\cdots\cdots$$
$$\cdots\cdots F_x(m\xi_o+\ell\xi_{-1}+\cdots+\xi_{-N+1})\,F_x(n\xi_o+m\xi_{-1}+\ell\xi_{-2}+\cdots+\xi_{-N})$$
$$F_x(n\xi_{-1}+m\xi_{-2}+\cdots+a\xi_{-N})\cdots\cdots$$
$$\cdots\cdots F_x(n\xi_{-N+1}+m\xi_{-N})\,F_x(n\xi_{-N}). \qquad (16)$$

The moments of the output signal depend only on the moments of the input; an output moment of a certain order depends only on the input moments of that order and lower. This can be seen by taking derivatives of equation (16).

Infinite-Order Outputs and Markov Processes

Suppose that the system shown in Figure 1 is driven by a source of random first-order samples. The output will be an infinite-order process according to the conventions used above because the impulse response has an infinite number of samples. In principle, the output statistics could be represented by equation (16). Actually this is a special kind of infinite-order process, generated by an impulse response which is a geometric-series, and can be represented more simply as a first-order Markov process.

The output of the filter of Figure 1 can be thought of as a linear combination of an infinite number of past inputs. The output can also be expressed as the sum of the present input plus one half of the previous input. In principle, an infinite number of output samples in a row are statistically related, but for this kind of situation, knowledge of the statistical relation between just two output samples in a row is sufficient to establish that between any number in a row. This follows because knowledge of the previous output completely summarizes past history in its effect on the next output sample. If the input were a second-order process, knowledge of the past two output samples would help in predicting the next output sample, since it would then be possible to compute the previous input sample and this would help in predicting the next input sample.

Let the input to a linear filter be first order. If the linear filter has finite memory, the joint relation between a number of consecutive output samples one greater than the number of memory states is necessary to completely describe the output process. The joint relation between more samples in a row can be derived, but will show that statistical dependence is only carried over the above-mentioned number of samples. If the linear filter has infinite memory (impulse response is a geometric sequence), the joint relation between two adjacent output samples enables one to derive the entire output process, which will show that statistical connection will extend (in principle) over an infinite number of output samples. If the impulse response is the sum of two exponentials, the output process is second-order Markov, and the joint relation between three consecutive samples is necessary to get a complete description of the output process.

Getting a higher-dimensional distribution of a Markov process is best done by leaving the c.f. domain as in the equations (17).

$$W(r_0, r_{-1}, r_{-2}) = W(r_0, r_{-1})\, w(r_{-2} | r_{-1}) \tag{17}$$

$$= W(r_0 - r_{-1}) \frac{w(r_{-1}, r_{-2})}{\int_{-\infty}^{\infty} w(r_{-1}, r_{-2})\, dr_{-2}}$$

Return now to the block diagram of Figure 1. Instead of fixing the gain in the feedback link at 1/2, let it have the value "a". The impulse response will have a sequence of values 1, a, a^2, a^3,.... Figure 7 shows the construction for the calculation of $F_{r_0, r_{-1}}(\xi_0, \xi_{-1})$ of the infinite-order output.

It follows from Figure 7 that the c.f. of Σ is equation (18).

$$F_\Sigma(\xi) = F_x(k_0\xi) F_x(ak_0\xi + k_{-1}\xi) F_x(a^2k_0\xi + ak_{-1}\xi)\ldots \tag{18}$$

Therefore, the desired two-dimensional output c.f. is (19).

$$F_{r_0, r_{-1}}(\xi_0, \xi_{-1})$$

$$= F_x(\xi_0) F_x(a\xi_0 + \xi_{-1}) F_x(a^2\xi_0 + a\xi_{-1}) F_x(a^3\xi_0 + a^3\xi_0 + a^2\xi_{-1})\ldots$$

$$= F_x(\xi_0) \prod_{n=1}^{\infty} F_x(a^n\xi_0 + a^{n-1}\xi_{-1}) \tag{19}$$

For example, let the input again be first-order Gaussian, with c.f. $= F_x(\xi) = e^{-\frac{\xi^2\sigma^2}{2}}$

From equation (19),

$$F_{r_0, r_{-1}}(\xi_0, \xi_{-1})$$

$$= e^{-\frac{\xi_0^2\sigma^2}{2}} \prod_{n=1}^{\infty} e^{-\frac{(a^n\xi_0 + a^{n-1}\xi_{-1})^2\sigma^2}{2}}$$

$$= e^{-\frac{\sigma^2}{2}\left[\xi_0^2 + \sum_{n=1}^{\infty}(a^{2n}\xi_0^2 + a^{2n-2}\xi_{-1}^2 + 2a^{2n-1}\xi_0\xi_{-1})\right]} \tag{20}$$

$$= e^{-\frac{\sigma^2}{2}\left(\frac{1}{1-a^2}\right)(\xi_0^2 + \xi_{-1}^2 + 2a\xi_0\xi_{-1})} \tag{20}$$

This gives the correct mean square of the output, $\left(\frac{\sigma^2}{1-a^2}\right)$, and a correlation coefficient between two adjacent samples of "a".

Derivation of Output Characteristic Functions for Higher-Order Inputs

So far only the response of linear sampled-data systems to first-order inputs has been considered. When the input signal is a higher-order process, it can sometimes be recognized as the result of a first-order process having been linearly filtered. Combine the cascaded filters into a single filter which is driven by a first-order input. When such a recognition cannot be made, more involved techniques are required. The form of these techniques will be indicated by the derivation of the three dimensional c.f. of a one-memory-state filter driven by a second-order input.

The joint relation between three output samples in a row will again be evaluated by forming their linear combinations. The appropriate construction is shown in Figure 8.

The input is second-order having a c.f. of $F_{x_0, x_{-1}}(\xi_0, \xi_{-1})$. The problem is to form the c.f. of the sum Σ, whose ingredients are now statistically related. Although only two inputs in a row are related, it is convenient here to represent the input process as fourth order having the c.f. $F_{x_0, x_{-1}, x_{-2}, x_{-3}}(\xi_0, \xi_{-1}, \xi_{-2}, \xi_{-3})$. This c.f. is the four-dimensional transform of the input distribution density (21).

$$F_{x_0, x_{-1}, x_{-2}, x_{-3}}(\xi_0, \xi_{-1}, \xi_{-2}, \xi_{-3}) \tag{21}$$

$$= \iiiint_{-\infty}^{\infty} w(x_0, x_{-1}, x_{-2}, x_3)\, e^{-j(\xi_0 x_0 + \xi_{-1} x_{-1} + \xi_{-2} x_{-2} + \xi_{-3} x_{-3})}\, dx_0\, dx_{-1}\, dx_{-2}\, dx_{-3}$$

The c.f. of the sum $x_0 + x_{-1} + x_{-2} + x_{-3}$ is equation (22).

$$F_{(x_0 + x_{-1} + x_{-2} + x_{-3})}(\xi)$$

$$= \iiiint_{-\infty}^{\infty} w(x_0, x_{-1}, x_{-2}, x_{-3})\, e^{-j\xi(x_0 + x_{-1} + x_{-2} + x_{-3})}\, dx_0\, dx_{-1}\, dx_{-2}\, dx_{-3} \tag{22}$$

$$= F_{x_0, x_{-1}, x_{-2}, x_{-3}}[\xi, \xi, \xi, \xi]$$

The c.f. of the sum Σ in Figure 8 is equation (23).

$$F_{\Sigma}(\xi) = F_{x_0, x_{-1}, x_{-2}, x_{-3}}\left[h_0\xi, (ah_0+h_{-1})\xi, (ah_{-1}+h_{-2})\xi, (ah_{-2})\xi\right] \tag{23}$$

The desired result is obtained from (23) in the usual fashion as (24),

$$F_{n_0, n_{-1}, n_{-2}}(\xi_0, \xi_{-1}, \xi_{-2}) = F_{x_0, x_{-1}, x_{-2}, x_{-3}}\left[\xi_0, (a\xi_0+\xi_{-1}), (a\xi_{-1}+\xi_{-2}), (a)\xi_{-2}\right] \tag{24}$$

Equation (24) gives some very simple results when applied to the problem of the propagation of second-order Gaussian statistics in the above filter. Let the input have the c.f. of equation (25). Its mean square is σ^2, and the correlation between adjacent input samples is σ_{12}.

$$F_{x_0, x_{-1}}(\xi_0, \xi_{-1}) = e^{-1/2\left[\sigma^2(\xi_0^2+\xi_{-1}^2)+2\sigma_{12}(\xi_0\xi_{-1})\right]} \tag{25}$$

Equation (26) gives the degenerate fourth-order c.f. of the second-order input process.

$$F_{x_0, x_{-1}, x_{-2}, x_{-3}}(\xi_0, \xi_{-1}, \xi_{-2}, \xi_{-3}) = e^{-1/2\left[\sigma^2(\xi_0^2+\xi_{-1}^2+\xi_{-2}^2+\xi_{-3}^2)+2\sigma_{12}(\xi_0\xi_{-1}+\xi_{-1}\xi_{-2}+\xi_{-2}\xi_{-3})\right]} \tag{26}$$

As dictated by equation (24), the desired three-dimensional output c.f. (equation (28)) is obtained making the substitutions (27) to the right-hand side of (26).

$$\left.\begin{aligned} \xi_0 &\longrightarrow \xi_0 \\ \xi_{-1} &\longrightarrow [a\xi_0+\xi_{-1}] \\ \xi_{-2} &\longrightarrow [a\xi_{-1}+\xi_{-2}] \\ \xi_{-3} &\longrightarrow a\xi_{-2} \end{aligned}\right\} \tag{27}$$

$$\begin{aligned} F_{n_0, n_{-1}, n_{-2}}(\xi_0, \xi_{-1}, \xi_{-2}) &= e^{-1/2\left\{\sigma^2\left[\xi_0^2+(a\xi_0+\xi_{-1})^2+(a\xi_{-1}+\xi_{-2})^2+a^2\xi_{-2}^2\right] + 2\sigma_{12}\left[\xi_0(a\xi_0+\xi_{-1})+(a\xi_0+\xi_{-1})(a\xi_{-1}+\xi_{-2})+a\xi_{-2}(a\xi_{-1}+\xi_{-2})\right]\right\}} \\ &= e^{-1/2\left\{(\xi_0^2+\xi_{-1}^2+\xi_{-2}^2)(\sigma^2+a\sigma^2+2a\sigma_{12}) + 2(\sigma^2+\sigma_{12}a+a^2\sigma_{12})(\xi_0\xi_{-1}+\xi_{-1}\xi_{-2})+2a\xi_0\xi_{-2}\sigma_{12}\right\}} \end{aligned} \tag{28}$$

The mean square of the third-order output signal is ($\sigma^2 + a\sigma^2 + 2a\sigma_{12}$). The correlation between adjacent samples is ($\sigma^2 + \sigma_{12}a + \sigma_{12}a^2$) and the correlation between every other sample is $\sigma_{12}a$.

In this paper, several kinds of random input signals were applied to various kinds of linear-sampled data filters. The means of getting complete statistical descriptions of the output signals were given. Except for certain kinds of input statistics, among them the Gaussian, the general explicit forms turn out to be rather unwieldy to apply although they are very easy to obtain.

The relationship between input statistics and output statistics has not turned out to be in the nature of a linear "transform function." The advantages of a transfer function are obtained however, since the output c.f. can always be written down directly when the input c.f. is given. The characteristics of the linear filter were completely determined by the joint input-output c.f., for an arbitrary input c.f. This joint c.f. has not been derived, but is derivable by the techniques of this paper.

The results of this paper make available to the engineer a lot more statistical information than he has been accustomed to using in system design and analysis, ie, more than mean square and autocorrelation function. It is hoped that he will learn to use these things to get more value from statistical system analysis.

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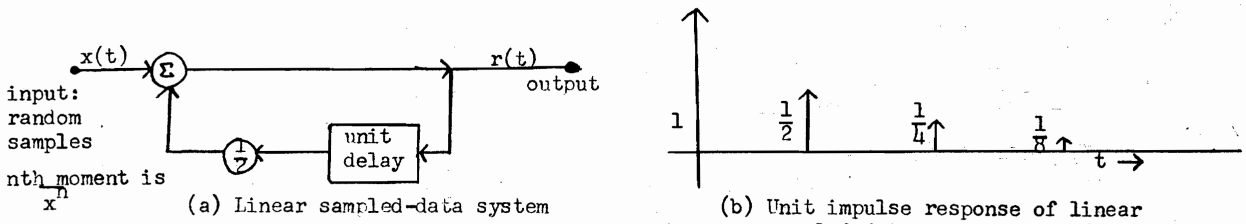
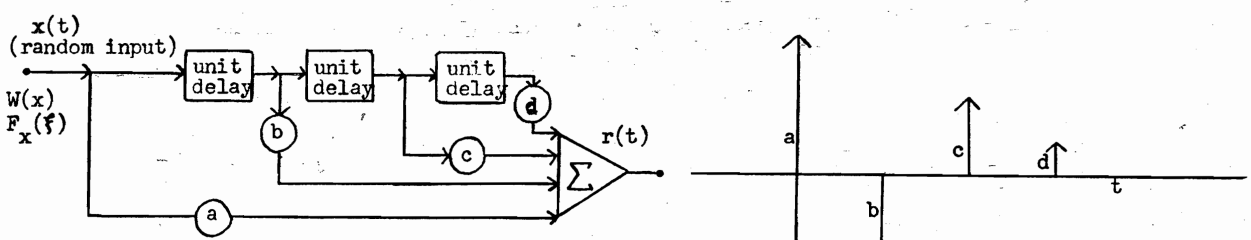


(a) Linear sampled-data system

(b) Unit impulse response of linear sampled-data system

Figure 1. A Linear Sampled-Data System Driven by Random Samples



(a) Finite-memory sampled-data system

(b) Impulse response of system

Figure 2. Sampled-Data System of Finite Memory Driven by Random Input

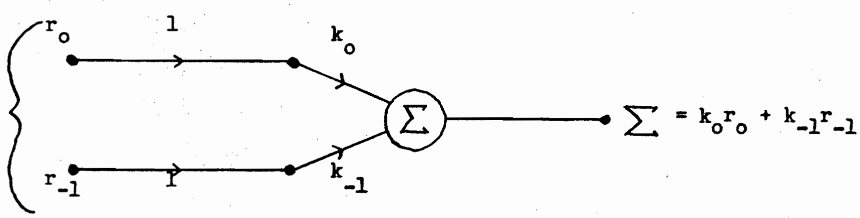


Figure 3. Forming Linear Combinations of Two Jointly-Related Random Variables

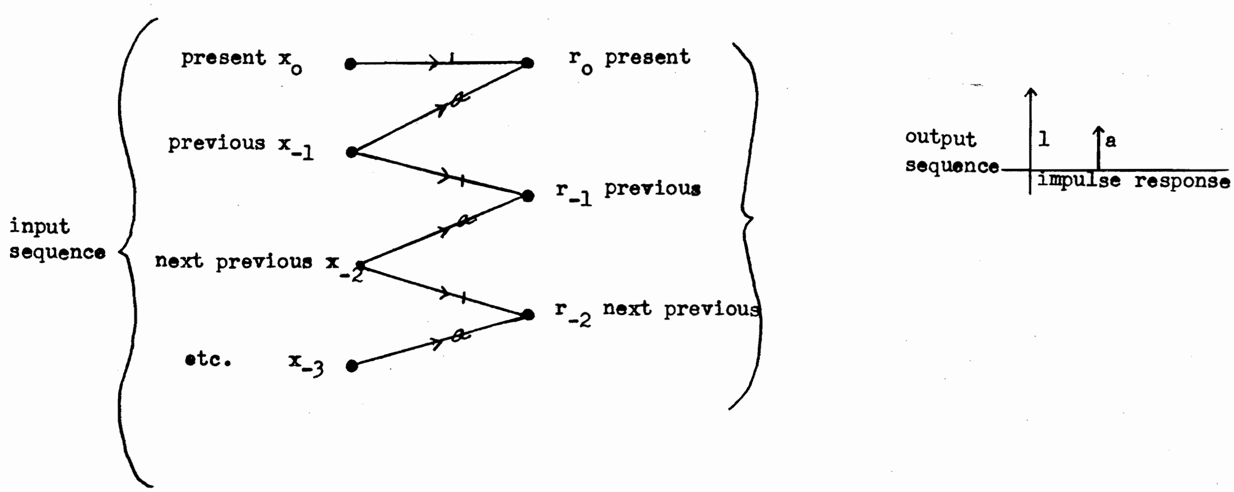


Figure 4. Combinational Representation of a Sampled-Data System

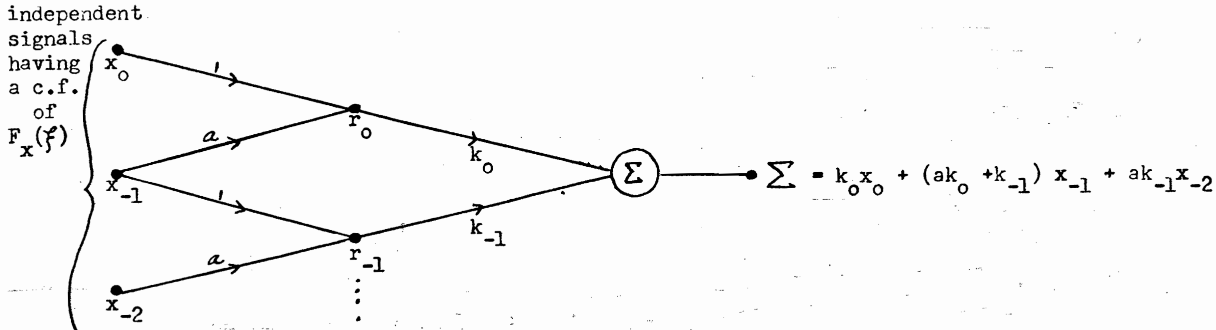


Figure 5. Construction for the Calculation of $F_{r_0, r_{-1}}(\xi_0, \xi_{-1})$

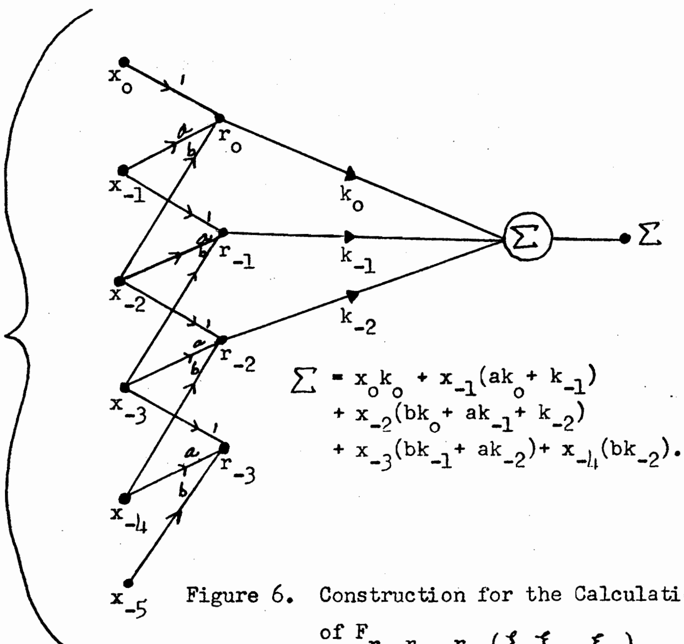


Figure 6. Construction for the Calculation of $F_{r_0, r_{-1}, r_{-2}}(\xi_0, \xi_{-1}, \xi_{-2})$

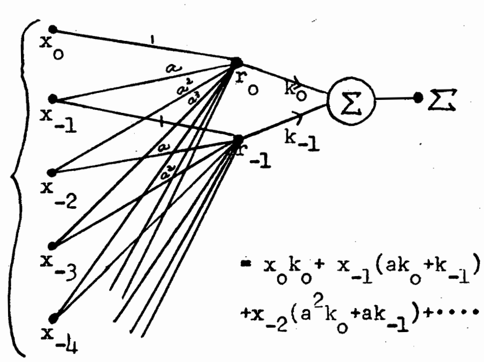


Figure 7. Construction for the Calculation of $F_{r_0, r_{-1}}(\xi_0, \xi_{-1})$

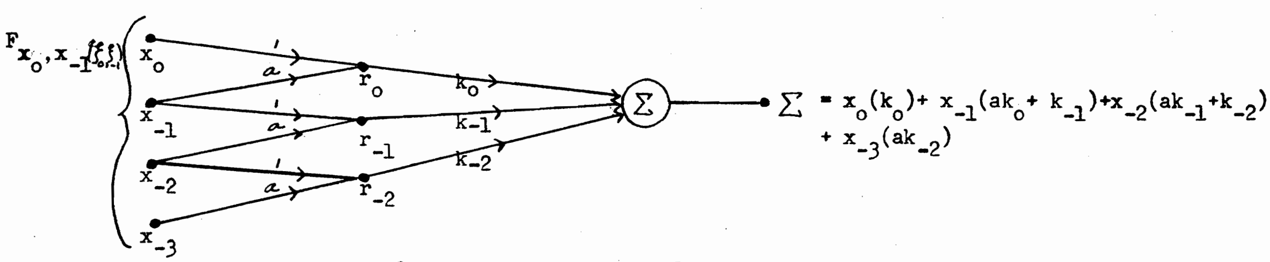


Figure 8. Construction for Calculation of Input c.f. when Output is a Second-Order Process.