Analysis of Amplitude-Quantized Sampled-Data Systems

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QUANTIZATION, or round-off, takes place whenever physical quantities are represented numerically. A quantizer is an operator that assigns a value to a variable equal to its closest integer. Analog-to-digital conversion consists of both sampling and quantizing.

Quantization noise is bounded between $\pm 1/2$ a quantum unit, and can be made small by choice of the basic unit. The smaller the unit, however, the more digits are required to represent the same physical quantities, and the greater is the difficulty and expense in storing and processing these quantities. To establish a balance between accuracy and economy, it is necessary to have a means of evaluating the distortion resulting from rough quantization. The analysis of quantization developed in this article is a statistical one. Although a quantizer is a nonlinear operator to signals, it acts linearly upon their probability distributions.

The probability density distribution of a quantizer output signal is discrete, consisting of a series of uniformly separated impulses, with spacing equal to the quantization box q. This density has a characteristic function (Fourier transform) which is periodic with a "frequency" $\varphi = 2\pi/q$. A comparison of quantization with the addition of a statistically independent noise uniformly distributed (between plus and minus $q/2$) shows that the quantizer output distribution density consists of samples of the distribution density of signal plus noise. Satisfaction of a quantizing theorem (like the sampling theorem for signals) assures that the statistics of a process can be recovered from statistics of its quantized samples.

Quantization noise is causally related to the quantizer input signal, yet in many respects, it behaves as if it were independent. When the quantizing theorem is satisfied, quantizing noise turns out to be precisely first-order flat-topped distributed, and is uncorrelated with the signal. The quantization of high-order (correlated) signals compares with the addition of first-order noise (statistically independent, white). When a multidimensional quantizing theorem is satisfied, quantization noises are first-order, even though signals may be highly correlated.

Statistical systems analyses are especially simple when the quantizing theorem is satisfied. The first step in any analysis is therefore a check for the extent of the satisfaction of the quantizing theorem. This has been done for the particular case of Gaussian-distributed signals, and the results give a general indication of how rough quantization can be, and yet the quantizing theorem may be considered satisfied to a good approximation. With a quantization box size q as large as three standard deviations, there will be only a 10% error in the theoretical mean square of the quantization noise, and an input which is 80% correlated will yield a quantization noise which is only 30% correlated (rather than completely uncorrelated). This represents very rough quantization, where two quantum levels cover a range of six standard deviations. Moments and densities can therefore be recovered to a good approximation when process samples are quantized to be either positive or negative and this has been demonstrated experimentally. Satisfaction of the quantizing theorem is not a stringent condition in most practical situations.

When the quantizing theorem is satisfied, the quantizer may be replaced by a source of independent first-order uncorrelated noise having zero mean and a mean square of $q^2/12$. Several examples are given to illustrate how this fact is used in the recovery of moments, distribution densities, and correlation functions from roughly quantized data. Examples are given to show how a quantized feedback system may be analyzed.

A "linear" quantizer system has an output consisting of the sum of two components. The first component is the same as the output of the linear equivalent (identical system except that the quantizers are replaced by unit gains) when excited by the given input. The second component is described statistically as the output of the linear equivalent, having no input signal, with the quantizer replaced by a source of uniformly-distributed independent first-order noise. The noise output and the characteristics of the linear equivalent system will both be independent of the signal and will completely characterize the situation for the large class of inputs which satisfy the quantizing theorem.

An external "dither" can frequently be added to the input of a quantizer feedback system to improve performance. The dither acts like a catalyst in insuring satisfaction of the quantizing theorem, and is filtered out by the feedforward member before reaching the output point. Sinusoidal, random Gaussian, or other independent signals may be injected to achieve this effect irrespective of the characteristics of the actual input signal. The effects of "dead zone" can be eliminated, systems can be "statistically linearized," and low-frequency limit cycles can be elimininated. Injection of an external dither can convert a very crude control system, one containing rough quantization, saturating quantization, or even hysteresis (as in contactor systems) to a beautifully linear, almost noise-free control system.

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