Reprinted from PROCEEDINGS OF THE IEEE INTERNATIONAL SYMPOSIUM ON CIRCUITS AND SYSTEMS, Philadelphia, Pennsylvania, May 4-7, 1987

A FUNDAMENTAL RELATIONSHIP BETWEEN THE LMS ALGORITHM AND THE DISCRETE FOURIER TRANSFORM

Bernard Widrow

Department of Electrical Engineering
Stanford University

ABSTRACT

The digital Fourier transform (DFT) and the adaptive least mean square (LMS) algorithm have existed for some time. This paper establishes a connection between them. The result is the "LMS spectrum analyzer," a new means for the calculation of the DFT. The method uses a set of N periodic complex phasors whose frequencies are equally spaced from DC to the sampling frequency. The phasors are weighted, then they are summed to generate a reconstructed signal. Weights are adapted to realize a best least squares fit between this reconstructed signal and the input signal whose spectrum is to be estimated. The magnitude squares of the weights correspond to the power spectrum.

For a proper choice of adaptation speed, the LMS spectrum analyzer will provide an exact N-sample DFT. New DFT outputs will then be available in steady-flow after introduction of each new data sample.

INTRODUCTION

During the past several years, work has been ongoing in the Electrical Engineering at Stanford University to determine the applicability of adaptive signal processing algorithms to problems in spectral analysis. Since Fourier techniques are in themselves least squares methods, one could believe that least squares adaptive algorithms might be somehow connected to Fourier analysis. Such a connection has been found. It is possible to compute a signal's digital Fourier transform exactly by making use of the LMS adaptive algorithm [1,2,3]. The method of calculation leads to an "LMS spectrum analyzer."

The work reported in this paper has been abstracted from a more complete paper on the subject entitled "The LMS Spectrum Analyzer" by B. Widrow, P. Baudrenghien, M. Vetterli, and P. Titchener [4]. Our objective here is to explain the approach and to state the principal results of the complete paper.

AN LMS SPECTRUM ANALYZER FOR THE COMPUTATION OF THE DFT

The LMS spectrum analyser, an adaptive system that could be used in the calculation of the DFT, is shown in Fig. 1. In this system, the input signal d_j to be Fourier analyzed is sampled, and the time index is j. The sampling period is T, and the sampling frequency is

$$\Omega = \frac{2\pi}{T}$$

The input d_j could be real or complex. The weights $w_0, w_1, \ldots w_{N-1}$ will in general be complex. The same is true for the weighted sum y_j and for the error ϵ_j. The weights are adjusted or adapted in accord with the complex LMS algorithm of Widrow, McCool, and Ball [2].

$$W_{j+1} = W_j + 2\mu\epsilon_j\overline{X}_j \quad .$$

This algorithm minimizes the mean square of the complex error ϵ_j, i.e. minimizes the mean of the sum of the squares of its real and imaginary parts. The terms in the equation are defined as follows: W_j is the current complex weight vector. W_{j+1} is the next complex weight vector.

$$W_j \triangleq \begin{bmatrix} w_{0j} \\ w_{1j} \\ \cdot \\ \cdot \\ \cdot \\ w_{(N-1)j} \end{bmatrix} .$$

The complex error ϵ_j is given by

$$\epsilon_j = d_j - y_j \ , \qquad \text{where}$$

$$y_j = X_j^T W_j \ , \qquad \text{and where}$$

$$X_j \triangleq \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{N}j} \\ \cdot \\ \cdot \\ \cdot \\ e^{i\frac{2\pi(N-1)}{N}j} \end{bmatrix} .$$

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Note that $\overline{X}_j$ is the conjugate of X_j and that $i = \sqrt{-1}$. The result of the adaptive process is W_j, which will be associated with the DFT of d_j.

The choice of the phasor components of the X_j-vector can be explained in the following way. We have used a total of N basis frequencies, including zero frequency. The fundamental basis frequency is Ω/N. The entire set of basis frequencies span the frequency range from zero up to the sampling frequency Ω. The fundamental phasor, a time function expressed in terms of the discrete time index j, is

$$e^{i\frac{\Omega}{N}jT} .$$

Since $\Omega T = 2\pi$, this can be written as

$$e^{i\frac{\Omega}{N}jT} = e^{i\frac{2\pi}{N}j} .$$

All of the components of the X-vector are powers of the above. A normalization factor $1/\sqrt{N}$ has been included to simplify the analysis of the system of Fig. 1. It causes the X_j-vector to have unit power.

The complex LMS algorithm is used to obtain the weights. The weight iteration formula can be expressed as

$$\begin{aligned} W_{j+1} &= W_j + 2\mu\epsilon_j\overline{X}_j \\ &= W_j + 2\mu\overline{X}j(d_j - y_j) \\ &= W_j + 2\mu\overline{X}_j(d_j - X_j^T W_j) \\ &= W_j + 2\mu d_j\overline{X}_j - 2\mu\overline{X}_j X_j^T W_j \\ &= (I - 2\mu\overline{X}_j X_j^T)W_j + 2\mu d_j\overline{X}_j \quad . \end{aligned}$$

Let the initial weight vector be set to zero. On a step by step basis, the weight vector versus time can be induced. Using the formula above,

$$\begin{aligned} W_1 &= (I - 2\mu\overline{X}_o X_o^T)\, W_o + 2\mu d_o\overline{X}_o \\ &= (I - 2\mu\overline{X}_o X_o^T)\, 0 + 2\mu d_o\overline{X}_o \\ &= 2\mu d_o\overline{X}_o \quad . \end{aligned}$$

Next,

$$\begin{aligned} W_2 &= (I - 2\mu\overline{X}_1 X_1^T)W_1 + 2\mu d_1\overline{X}_1 \\ &= 2\mu d_o\overline{X}_o - 4\mu^2\overline{X}_1 X_1^T\overline{X}_o d_o + 2\mu d_1\overline{X}_1 \\ &= 2\mu(d_o\overline{X}_o + d_1\overline{X}_1) - 4\mu^2\overline{X}_1(X_1^T\overline{X}_o)d_o \quad . \end{aligned}$$

This can be simplified to

$$W_2 = 2\mu(d_o\overline{X}_o + d_1\overline{X}_1) \quad .$$

Next,

$$\begin{aligned} W_3 &= (I - 2\mu\overline{X}_2 X_2^T)W_2 + 2\mu d_2\overline{X}_2 \\ &= 2\mu(d_o\overline{X}_o + d_1\overline{X}_1 + d_2\overline{X}_2) - 4\mu^2\overline{X}_2(X_2^T\overline{X}_o d_o + X_2^T\overline{X}_1 d_1) \end{aligned}$$

This can be simplified to

$$W_3 = 2\mu(d_o\overline{X}_o + d_1\overline{X}_1 + d_2\overline{X}_2) \quad .$$

We can easily generalize this result:

$$W_j = 2\mu \sum_{m=0}^{j-1} d_m\overline{X}_m \ , \qquad j=1, \dots, N \quad .$$

An interesting case is that for j = N. From this relation we obtain

$$W_N = \frac{2\mu}{\sqrt{N}} \begin{bmatrix} \sum_{m=0}^{N-1} d_m \\ \sum_{m=0}^{N-1} d_m e^{-i\frac{2\pi}{N}m} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{m=0}^{N-1} d_m e^{-i\frac{2\pi(N-1)}{N}m} \end{bmatrix} .$$

Except for the scale factor, it is clear from the above that the elements of W_N comprise the values of the DFT of d_j over the uniform time window from j = 0 to j = N-1.

This formula is based on orthogonality of X-vectors at different times and it applies for $1 \leq j \leq N$. Beyond this range, we need a new formula since, for example, X_0 is identical to and of course not orthogonal to X_N.

With some further algebraic work along the same lines, a completely general formula for W_j can be derived which would be applicable over all $j \geq 1$, assuming the initial condition $W_0 = 0$. The result is

$$\begin{aligned} W_j = {} & 2\mu \sum_{m=j-N}^{j-1} d_m\overline{X}_m \\ & + 2\mu(1-2\mu) \sum_{m=j-2N}^{j-N-1} d_m\overline{X}_m \\ & + 2\mu(1-2\mu)^2 \sum_{m=j-3N}^{j-2N-1} d_m\overline{X}_m \\ & + 2\mu(1-2\mu)^3 \sum_{m=j-4N}^{j-3N-1} d_m\overline{X}_m \\ & \quad \cdot \\ & \quad \cdot \\ & \quad \cdot \end{aligned}$$

In using this formula, it is understood that the allowed ranges of the index m for each of the sums is set by the upper and lower limits unless these limits are negative. The ranges of m must first be

$m \geq 0$, then determined by the sum's limits. Thus in applying the formula, one sees that for $N \geq j \geq 0$ only the first term in the series exists, all the rest are zero, and the first term agrees as it should with the previously derived formula for $j = N$.

A critical choice of μ is the value $\mu = \frac{1}{2}$. Making this choice, the above series reduces to its first term regardless of the value of j. Let $\mu = \frac{1}{2}$ and let j be any integer multiple of N. W_j will be proportional to the DFT of the previous N samples of d_j. Thus, at time ℓN, the formula for W_j becomes

$$W_{lN} = \sum_{m=lN-N}^{lN-1} d_m \overline{X}_m$$

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} \sum_{m=lN-N}^{lN-1} d_m \\ \sum_{m=lN-N}^{lN-1} d_m e^{-i\frac{2\pi}{N}m} \\ \vdots \\ \sum_{m=lN-N}^{lN-1} d_m e^{-i\frac{2\pi(N-1)}{N}m} \end{bmatrix} .$$

It is evident from this expression that W_{lN} is indeed proportional to the above stated DFT.

CONCLUSION

We have shown that the LMS algorithm can be used to calculate the DFT. After each block of N data samples are analyzed, the components of the adaptive weight vector comprise the respective frequency components of the DFT. It is shown in [4] how the system of Fig. 1 can be used to calculate the DFT in steady flow, giving an instantaneous DFT every sampling period.

The digital Fourier transform is only an approximation to the true Fourier transform. The LMS algorithm in turn only approximates a true least squares solution. However when the weights in Fig. 1 are driven by LMS with $\mu = 1/2$, the DFT of the input signal d_j is given exactly by the LMS weights, no approximation is involved. Evidently, the approximations inherent in the DFT are matched by the approximation inherent in the LMS algorithm.

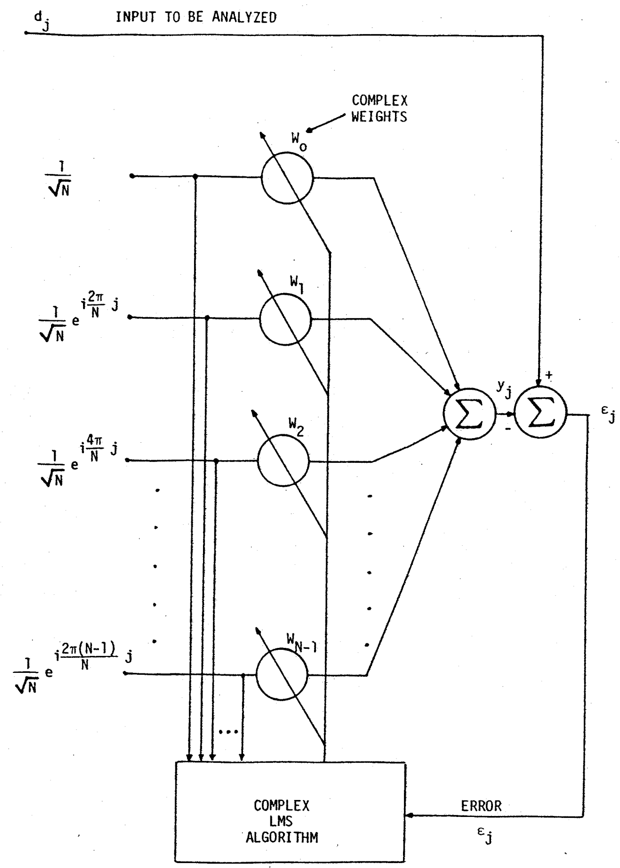

Fig. 1 The LMS Spectrum Analyzer

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